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## The equity premium puzzle and the resolution by 'myopic loss aversion' revisited

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# 1. Introduction

There is a large discrepancy between the returns on stocks, bonds, and cash. Stocks have outperformed bonds and cash by a large margin for a long period in combination with low cash returns. This fact is referred to as the *equity premium puzzle*. Mehra and Prescott [6] showed that the puzzle could not be resolved with plausible levels of risk aversion in the content of power utility functions.

Benartzi and Thaler [1] claimed that the equity premium puzzle could be resolved by assuming that investors suffer from *myopic loss aversion*. Myopic loss aversion is a combination of the tendency to weigh losses heavier than gains<sup>1</sup> and a high frequency of portfolio evaluation. Frequent evaluation implies that an investor evaluates investments according to his (short) evaluation period, even if his investment horizon is (much) longer. Analyzing 1926 - 1990 data, Benartzi and Thaler<sup>2</sup> estimated the evaluation period for the United States to be roughly one year.

We investigate the variability of evaluation periods across countries. We focus on the 1978 - 1994 period for 4 countries: the United States, the United Kingdom, Germany, and Japan.<sup>3</sup> Furthermore, we investigate the stability of the evaluation periods within countries: how sensitive are evaluation periods to changes in data and parameters? Like B&T, we use the cumulative version of prospect theory to evaluate different investments.

The remainder of this paper is organized as follows. Section 2 describes the model used: the cumulative version of prospect theory. Section 3 provides an implementation of the model on the equity premium puzzle and the set-up of B&T's study. Section 4 shows the results of our analysis. Section 5 contains the discussion, while section 6 summarizes and concludes.

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<sup>1</sup> See Tversky and Kahneman [10].

<sup>2</sup> In the remainder of this paper we will refer to Benartzi and Thaler [1] as B&T.

<sup>3</sup> In the remainder of the paper we use the following abbreviations: USA for the United States, UK for the United Kingdom, GER for Germany, and JPN for Japan.

## 2. The model

The starting point of our analysis is an average individual investor<sup>4</sup> using the cumulative version of prospect theory<sup>5</sup> to evaluate his investment opportunities. As a consequence, the investor's decisions only depend on returns (see below).

In the classical expected utility theory, the utility of a financial product with cash flows contingent on events and the passage of time is the sum of the utilities of the outcomes, each weighted by its probability. Standard theories of finance maintain that economic subjects are indifferent among frames of cash flows: when a package of cash flows is split up and rebundled without affecting the net cash flow, the value of the package remains unchanged. Contrary to these theories, descriptive theories of choice behavior, such as prospect theory, do not assume frame invariance. Empirical evidence<sup>6</sup> confirms the rejection of the assumption of frame invariance.

Prospect theory makes two major modifications to expected utility theory as it is usually expounded: (i) prospects are evaluated according to gains and losses, not final assets or wealth; and (ii) the value of each outcome is multiplied by a decision weight, not by an additive probability. So, the prospective value of a prospect which pays off  $x_i$  with probability  $p_i$  is given by

$$V(\cdot) = \sum \pi_i \cdot v(x_i), \quad (1)$$

where  $\pi_i$  is the decision weight associated with outcome  $i$  and  $v(x_i)$  is the prospective value of  $x_i$ .

Prospect theory investors, as opposed to expected utility investors, evaluate their choices with an S-shaped value function and two inverse S-shaped weighting functions for the probabilities of the outcomes. The two parts of the value function imply risk-averse behavior in the domain of gains and risk-seeking behavior in the domain of losses. The value function is normalized by a reference point; the value of the status quo is zero and gains and losses

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<sup>4</sup> In the terminology of Brennan [2], we focus on an average individual investor rather than on a representative investor.

<sup>5</sup> See Tversky and Kahneman [10].

<sup>6</sup> See, e.g., Tversky and Kahneman [10] or Quiggin [7].

are relative to this reference point. Tversky and Kahneman [10] provided a three-parameter expression for the value function and parameter estimates:

$$v(x) = \begin{cases} x^\alpha & \text{if } x \geq 0 \\ -\lambda \cdot (-x)^\beta & \text{if } x < 0. \end{cases} \quad (2)$$

The parameter  $\lambda$  captures empirically observed *loss aversion* and is estimated as 2.25 while  $\alpha$  and  $\beta$  are estimated as 0.88.<sup>7</sup> B&T used these values as well. Figure 1 below shows the prospect theory value function for these parameter estimates.

**FIGURE 1: HERE**

It should be noted that, given this particular value function with  $\alpha = \beta$ , an investor's preference order is independent of the fact whether net cash flows or rates of return are used. The ratio of the prospective value of the different positions, and thus the preference order, does not change because of the form of the prospective value function.

Cumulative prospect theory uses decision weights that depend on the objective probability of an outcome and the rank-order of all outcomes.<sup>8</sup> Furthermore, the weights are different for outcomes that are framed as gains and those that are framed as losses. The inverse S-shape implies that small probabilities are overweighted while moderate and high probabilities are underweighted. Figure 2 below shows these weighting functions. The less curved weighting function refers to losses and the more curved weighting function refers to gains.

**FIGURE 2: HERE**

### 3. Implementation

A feature that will be very important in the remainder of this paper is the *evaluation period*. This feature can be explained most easily by the example B&T borrowed from Samuelson

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<sup>7</sup> If  $v(x) = x^\alpha$ , then  $\alpha$  equals 1 minus the coefficient of relative risk aversion in expected utility theory.

<sup>8</sup> The functional form of the weighting functions, the relation between the weighting function and the cumulative distribution function, and the relation between the weighting functions and the decision weights can be found in appendix 1.

[8] . The example rests on a 50-50 bet between winning \$200 and losing \$100. Samuelson asked some lunch colleagues to bet each \$200 to \$100 that the side of a coin *they* specified would not appear at the first toss. Most colleagues rejected an offer to play one bet, as would a prospect theory investor. However, one of Samuelson's colleagues would be willing to accept the offer to play one hundred bets sequentially. This was his explanation. "One toss is not enough to make it reasonably sure that the law of averages will turn out in my favor. But in a hundred tosses of a coin, the law of large numbers will make it a darn good bet. I am, so to speak, virtually sure to come out ahead in such a sequence, and that is why I accept the sequence while rejecting the single toss." As Samuelson [8] showed, this choice is irrational for an expected utility investor.<sup>9</sup> However, it is completely acceptable for a prospect theory investor.

In the first choice problem, the evaluation period is related to one bet. With such a short evaluation period, most people prefer less risk: do not gamble. According to one of Samuelson's colleagues: "I won't bet because I would feel the \$100 loss more than the \$200 gain. But I'll take you on if you promise to let me make 100 such bets." In this situation, the large probability (50 percent) of a loss weighs more heavily than the same probability of a (larger) gain. This results from loss aversion: losses loom larger than gains.

In the second choice problem, the evaluation period is related to a sequence of one hundred bets. With a longer evaluation period, most people are willing to take more risk. Choices are made according to the long term evaluation period risk, which is in this situation equal to 'one hundred bet risk'. Now, the (significantly) positive expected value of one bet entails that the probability of a loss after one hundred bets is very small. Therefore, (large) potential gains weigh more heavily than (small) potential losses.

So, prospect theory investors are willing to take more risk if they evaluate their portfolio less frequently. This implies that the attractiveness of asset classes will depend on the evaluation period of the investor. The longer the investor's evaluation period, the more attractive the risky asset with the higher expected return will be.

B&T investigated whether the equity premium puzzle could be resolved by assuming that

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<sup>9</sup> Samuelson's theorem, a short proof, and some related remarks can be found in appendix 2.

investors suffer from myopic loss aversion. They began by asking what combination of loss aversion and evaluation period would be necessary to explain the historical return pattern? They then asked how often would a prospect theory investor have to evaluate his portfolio in order to be indifferent between the historical distributions of stocks and bonds? Analyzing 1926 - 1990 data, B&T found that the evaluation period for the United States is roughly one year.

B&T's analysis rests on the assumption that the representative prospect theory investor is indifferent between stocks and bonds, given his  $k^*$ -month evaluation period. To elicit this evaluation period, B&T calculate the prospective value of  $k$ -month returns on stocks and bonds for  $k \in \{1, \dots, 18\}$ . The distribution of  $k$ -month returns is constructed by drawing a large number of  $k$  monthly returns from the empirical distribution function and multiplying them. The optimal estimate of the evaluation period,  $k^*$ , is that value of  $k$  for which the prospective value of stocks equals the prospective value of bonds.

## 4. Results

Our data set covers the period January 1978 until June 1994. We use monthly returns on stocks and bonds. These returns show no significant serial correlation, so  $k$ -month return distributions can be constructed as in B&T.<sup>10</sup>

Before we proceed, we want to give an indication of the magnitude of the returns on stocks and bonds and the stock-bond equity premiums during the period from January 1978 until June 1994.<sup>11</sup> Table 1 shows the mean of the monthly returns on stocks and bonds and the monthly equity premiums for this period.

The annualized stock-bond equity premiums vary from roughly 3.7 percent for Japan to 6.0 percent for Germany. To get an indication of the stability, we bootstrapped the stock-

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<sup>10</sup> The von Neumann ratio is insignificant for all countries at the 5 percent significance level. This implies that the hypothesis of no first-order serial correlation cannot be rejected and we can use the method described above to construct  $k$ -month returns. At the beginning of our study, we also investigated France and the Netherlands. However, in these countries the bond returns showed significant first-order serial correlation. Of course, an autoregressive model could be used to construct an appropriate return distribution.

<sup>11</sup> All returns and premiums are nominal.

|                | USA   | UK    | GER   | JPN   |
|----------------|-------|-------|-------|-------|
| stocks         | .0122 | .0153 | .0108 | .0108 |
| bonds          | .0085 | .0105 | .0060 | .0078 |
| equity premium | .0037 | .0047 | .0049 | .0030 |

Table 1: Monthly returns and equity premiums.

bond equity premiums 1000 times.<sup>12</sup> Table 2 shows the mean and standard deviation of the bootstrapped premiums as well as the lower and upper bounds of the 95 percent confidence interval for the stock-bond equity premiums.

Both the standard deviations and the confidence intervals show that the equity premiums are not stable. For all countries, the hypothesis that the equity premium is equal to zero, i.e., there is no equity premium, cannot be rejected. This instability of equity premiums may influence the subsequent analysis; if the equity premiums are instable, how stable are the evaluation periods?

Now, the first question we ask, as B&T did, is what evaluation period makes a prospect theory investor indifferent between holding his assets in stocks and bonds? To this end, we calculate 50,000  $k$ -month returns for stocks and bonds for all countries and the prospective value of these simulated return distributions. Table 3 shows the evaluation periods.

It is obvious from table 3 that evaluation periods are *not* constant across countries. For evaluation periods calculated with stocks and bonds, we find strikingly short evaluation periods for the United States and the United Kingdom while we find somewhat longer periods for Japan and Germany. However, all periods are shorter than the evaluation period of 12 months that B&T found for the United States in the period 1926 - 1990.

So, evaluation periods are not constant across countries. But, how stable is an evaluation period within one country? To this end, we calculate for each country 250 evaluation periods with bootstrapped returns. Per run, we used 10,000 simulated  $k$ -month returns to calculate the evaluation period. As could be expected by the instability of the equity premiums, table 4

<sup>12</sup> A short, clear explanation of the bootstrap can be found in Diaconis and Efron [3]. A more extensive explanation can be found in, e.g., Efron and Tibshirani [4].

|                    | USA    | UK     | GER    | JPN    |
|--------------------|--------|--------|--------|--------|
| mean               | .0038  | .0046  | .0047  | .0031  |
| standard deviation | .0032  | .0035  | .0037  | .0046  |
| 95% lower bound    | -.0028 | -.0023 | -.0024 | -.0060 |
| 95% upper bound    | .0097  | .0114  | .0115  | .0119  |

Table 2: Bootstrapped monthly stock-bond equity premiums.

shows that the evaluation periods are not stable. The quantiles show that there is significant variation in the widths of the confidence intervals.

Second we ask the question what mix is optimal given the estimated evaluation periods? Table 5 shows the optimal mix between stocks and bonds using the relevant evaluation period.

Table 5 shows that the optimal percentages differ from country to country.<sup>13</sup> Furthermore, all optimal stock percentages are smaller than or equal to fifty percent. So, even if the typical investor is indifferent between two asset classes, the optimal mix need not be fifty-fifty, as one perhaps naively would expect.

To summarize: B&T claimed that the equity premium puzzle could be resolved by assuming that investors suffer from myopic loss aversion. They found that the evaluation period for the United States is roughly one year and the optimal stock-bond portfolio has invested between 30 and 55 percent in stocks. For a 1978 - 1994 period, we investigate the variability of evaluation periods across countries for a different data set. Using the bootstrap method, we show that evaluation periods are not constant across countries. We find strikingly short evaluation periods for the United States and the United Kingdom, while we find somewhat longer periods for Japan and Germany. However, all periods are shorter than the evaluation period of 12 months that B&T found for the United States. Furthermore, evaluation periods are not stable within countries. We also find that the optimal percentages of the portfolio allocated to stocks differ from country to country: all percentages are smaller than or equal to fifty percent.

<sup>13</sup> Analogously to the result B&T found, the differences in prospective value around the optimal mix are not pronounced.



|                   | USA | UK | GER | JPN |
|-------------------|-----|----|-----|-----|
| evaluation period | 3   | 5  | 10  | 8   |

Table 3: Evaluation period per country (in months).

## 5. Discussion

### 5.1 *Simultaneous analysis of portfolio mix and evaluation period*

On the preceding pages, we stumbled upon a number of puzzling issues. The foremost among them is the procedure used by B&T to determine the evaluation period. This procedure consists of finding the period for which the average individual investor would be indifferent between stocks and bonds. B&T’s estimation of the evaluation period rests on the equality of the prospective value of stocks and bonds. Apparently, the average individual investor should be indifferent between investing an additional dollar in stocks and investing an additional dollar in bonds. In this setting, the *marginal* prospective value of stocks should equal the marginal prospective value of bonds. However, B&T equate the *absolute* prospective value of stocks and bonds. Why should the absolute prospective value be the same? Investors may *not* be indifferent between the assets and still wish to hold them *both*.

Because of the power utility like form of the value function,

$$v(x) = \begin{cases} x^\alpha & \text{if } x \geq 0 \\ -\lambda \cdot (-x)^\beta & \text{if } x < 0, \end{cases}$$

the ratio of the prospective value of stocks and the prospective value of bonds is independent of wealth invested. Therefore, the ratio of the prospective value of total wealth invested in stocks and the prospective value of total wealth invested in bonds equals the ratio of the prospective value of one additional dollar invested in stocks and the prospective value of one additional dollar invested in bonds: ‘marginal’ analysis is equivalent to ‘absolute’ analysis. Because the prospective value of an asset class only depends on the historical return distribution, the prospective value does not change if the average individual investor invests more (or less) in this asset class. This implies that if the prospective value of stocks is unequal to the prospective

|                 | USA  | UK   | GER  | JPN  |
|-----------------|------|------|------|------|
| mode            | 1    | 1    | > 12 | 1    |
| (if mode > 12)  |      |      | 4    |      |
| q <sub>5</sub>  | 0    | 1    | 2    | 0    |
| q <sub>10</sub> | 0    | 1    | 2    | 1    |
| q <sub>25</sub> | 1    | 1    | 4    | 1    |
| q <sub>50</sub> | 3    | 4    | 7    | 3    |
| q <sub>75</sub> | 12   | 12   | 12   | 7    |
| q <sub>90</sub> | > 12 | > 12 | > 12 | > 12 |
| q <sub>95</sub> | > 12 | > 12 | > 12 | > 12 |

Table 4: Statistics of stock-bond evaluation periods.

value of bonds, the average individual investor would want to invest all his wealth into the asset class with the highest prospective value.

In the analysis of the optimal mix between stocks and bonds, B&T focus on the prospective value of a *portfolio*, given the evaluation period. We feel one should analyze the portfolio mix and the evaluation period simultaneously. If we focus on a portfolio, the prospective value of investing one additional dollar in an asset class *does* change if the investor changes his asset allocation.

For all countries and evaluation periods ranging from 1 to 18 months, we calculated the prospective value of each portfolio between the all stocks and all bonds portfolio, in 5 percent increments. Figure 3 shows the contour plot for the United States.<sup>14</sup>

### FIGURE 3: HERE

The calculations also provide the relation between the percentage of the portfolio optimally allocated to stocks and the evaluation period. All contour plots show that the percentage rises if the evaluation period becomes longer. Figure 4 shows this relation for all countries.

<sup>14</sup> The plots for the United Kingdom, Germany, and Japan are roughly similar.

|          | USA | UK | GER | JPN |
|----------|-----|----|-----|-----|
| % stocks | 40  | 50 | 40  | 40  |
| % bonds  | 60  | 50 | 60  | 60  |

Table 5: Percentage stocks and bonds in the optimal mix.

#### FIGURE 4: HERE

It is clear that there is an almost linear relation between the optimal percentage of the portfolio allocated to stocks and the evaluation period. This suggests that the evaluation period could be estimated using the average asset mix of a country. B&T claim that the most frequent allocation between stocks and bonds for individuals in the United States is 50-50. This implies that the evaluation period of the United States should be 4 rather than 12 months. Table 6 shows the asset mix corresponding to an evaluation period of 12 months. The percentage allocated to stocks ranges from 70 percent for the United States and the United Kingdom and 50 percent for Germany to 45 percent for Japan.

## 5.2 *What drives the results?*

As noted above, myopic loss aversion shows three distinguishable features: an S-shaped value function, probability weighting, and an evaluation period. Both the S-shaped value function and the probability weighting are part of cumulative prospect theory. B&T claim that loss aversion, which is part of the S-shaped value function, is the main determinant of their result; the specific functional forms of the value function and weighting functions are not critical. When we take a *piecewise linear* value function, ignore probability weighting, and postulate that indices for stocks and bonds follow jointly a geometric Brownian motion, we find results qualitatively similar to those obtained using the B&T setup. In fact, the evaluation periods are even shorter.

For geometric Brownian motion, we can calculate the prospective value of stocks and bonds explicitly. For portfolios, we need an approximation (see appendix 3 for details). For parameter estimates from our 1978 - 1994 data set and a given value of  $\lambda$ , we find that the

|          | USA | UK | GER | JPN |
|----------|-----|----|-----|-----|
| % stocks | 70  | 70 | 50  | 45  |
| % bonds  | 30  | 30 | 50  | 55  |

Table 6: Asset mix corresponding to an evaluation period of 12 months.

expected value of the return on a 50-50, buy-and-hold portfolio is a monotonically increasing function of  $t$  for all countries.<sup>15</sup> Again, there is no global maximum for an evaluation period of roughly one year.

## 6. Summary and concluding remarks

Benartzi and Thaler provided the interesting hypothesis that myopic loss aversion could solve the equity premium puzzle. Our analysis focuses on the differences in and the stability of evaluation periods for different countries. We show that evaluation periods are neither constant across countries nor stable. Evaluation periods are strikingly sensitive to the data.

Benartzi and Thaler's hypothesis that investors use a relatively short evaluation period, even if they have a long investment horizon, is intuitively appealing. We feel that the analysis should focus on a simultaneous analysis of the evaluation period and the portfolio mix. Our analysis suggests that there is an almost linear relation between the optimal percentage of stocks in a stock-bond portfolio and the evaluation period. However, additional research is necessary concerning a simultaneous analysis of the asset mix, evaluation periods, and the method to evaluate alternatives.<sup>16</sup>

<sup>15</sup> It should be noted that for large values of  $\lambda$ , the expected value of the return on the portfolio is no longer monotonically increasing. This value decreases before it increases for longer evaluation periods.

<sup>16</sup> Forthcoming research contains an analysis of the optimal asset mix for a portfolio consisting of stocks, bonds, and a risk-free asset for different value functions. Of course, time (evaluation period) plays a major role in this analysis as well.

## Appendix 1

In prospect theory, decision weights are different for gains and losses. Furthermore, in the cumulative version of prospect theory, they depend on the cumulative distribution function,  $F(\cdot)$ , of the outcomes and not only on the probability of a single outcome. So, decision weight  $\pi_i$ , which is related to outcome  $x_i$ , depends on the sign of outcome  $x_i$  and the cumulative distribution function. In particular, the decision weight attached to a *negative* outcome  $x_i$  is:

$$\pi_i^- = w^-(F(x_i)) - w^-(F^*(x_i)) , \quad (3)$$

where  $w^-(\cdot)$  is the transformation function of the cumulative distribution function  $F(\cdot)$  and  $F^*(x_i)$  is defined as the probability of obtaining an outcome that is *strictly less* than  $x_i$ . Furthermore, the decision weight attached to a *positive* outcome  $x_i$  is:

$$\pi_i^+ = w^+(G(x_i)) - w^+(G^*(x_i)) , \quad (4)$$

where  $w^+(\cdot)$  is the transformation function related to positive outcomes,  $G(x_i)$  is defined as the probability of obtaining an outcome that is equal to or better than  $x_i$ , and  $G^*(x_i)$  is defined as the probability of obtaining an outcome that is strictly better than  $x_i$ .

Tversky and Kahneman [10] provided a one-parameter expression for these weighting functions:

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}} . \quad (5)$$

Furthermore, they provided parameter estimates that are different for gains and losses. The parameter,  $\gamma$ , is estimated as 0.61 and 0.69 for gains and losses, respectively. Figure 2 showed these weighting functions for cumulative probabilities.

## Appendix 2

Samuelson's theorem states that if someone is unwilling to accept a single play of a bet at any wealth level that could occur over the course of some number of repetitions of the bet, then accepting a sequence of such independent bets is inconsistent with expected utility theory. A short proof is the following. If for whatever value of  $Y$  :  $E[U(Y + \underline{X}_1)] < U(Y)$  and  $\underline{X}_1$  and  $\underline{X}_2$  are independent and identically distributed, then  $E[U(Y + \underline{X}_1 + \underline{X}_2)] = E[E[U(Y + \underline{X}_1 + \underline{X}_2) | \underline{X}_1]] < E[U(Y + \underline{X}_1)] < U(Y)$ . So, if one bet is unacceptable, two are even worse.

Incidentally, Samuelson [8] posits that what his colleague *really* had in mind was a sequence of one hundred bets each one hundredth as big as the original bet. Because a utility function can be approximated reasonably well with a linear function when the outcomes are sufficiently small, a utility maximizer locally shows risk neutral behavior. Therefore, a sequence of bets becomes acceptable if the individual bets are sufficiently small and the expected value is positive. In the case of Samuelson's colleague, the random outcome of the sequence of bets will cluster around the expected value of fifty with a relatively small standard deviation, so the compound bet will be acceptable to most expected utility maximizers.

Note that B&T's response to the colleague's reasoning is quite different. They do not speculate about what the colleague might have done if only he had thought about the proposition more carefully. They accept his behavior and rationalization, his 'mental accounting', and use the change of wealth relative to the status quo as a pivot for modelling actual rather than possibly desirable behavior. To strengthen the case for prospect theory, they refer to the 'extremely counterintuitive results' obtained by maximizing the expectation of utility functions that display constant relative risk aversion (CRRA): *An investor who wants mostly stocks in his portfolio at age 35 should still want the same allocation at age 64* (Benartzi and Thaler [1], p. 81). According to B&T, a prospect theory investor would never be so folly. Nor would, however, an investor with a utility function measuring utility relative to a minimum terminal wealth and displaying hyperbolic absolute risk aversion (HARA).

Samuelson [9] showed that with utility functions such as

$$U = \log (W - S)$$

or

$$U = \frac{(W - S)^\gamma}{\gamma},$$

where  $U$  is utility,  $W$  is terminal wealth,  $S > 0$  is minimum terminal wealth insisted on, and  $0 \neq \gamma < 1$  is the parameter of relative risk tolerance, the fraction rationally allocated to equities decreases with age toward retirement.

### Appendix 3

Analogously to B&T, we analyze a situation without probability weighting where the value function is replaced by a piecewise linear form with the same loss aversion parameter as before:

$$v(x) = \begin{cases} x & \text{if } x \geq 0 \\ \lambda \cdot x & \text{if } x < 0, \end{cases}$$

with  $\lambda > 1$ . This can be rewritten as

$$\begin{aligned} v(x) &= x \cdot \mathbf{I}_{[0, \infty)}(x) + \lambda x \cdot \mathbf{I}_{(-\infty, 0)}(x) \\ &= x + (\lambda - 1) \cdot \mathbf{I}_{(-\infty, 0)}(x). \end{aligned} \quad (6)$$

We assume that the price of an index  $x$  follows a geometric Brownian motion

$$dx = \mu_x x dt + \sigma_x x dz, \quad (7)$$

where  $dz$  is a Wiener process and  $\mu_x$  and  $\sigma_x$  are constant. The  $t$ -period return on the index  $x$  is given by

$$\begin{aligned} r_x(t) &= \frac{x_t - x_0}{x_0} \\ &= \exp\left(\left(\mu_x - \frac{1}{2}\sigma_x^2\right)t + \sigma_x dz_t\right) - 1 \\ &\equiv e^{y_t} - 1. \end{aligned} \quad (8)$$

The prospective value of this return is given by

$$\begin{aligned} \mathbb{E}[v(r_x(t))] &= \mathbb{E}[v(e^{y_t} - 1)] \\ &= \mathbb{E}[e^{y_t} - 1] + (\lambda - 1) \cdot \mathbb{E}[(e^{y_t} - 1) \cdot \mathbf{I}_{(-\infty, 0)}(e^{y_t} - 1)] \\ &= e^{\mu_t} - 1 + (\lambda - 1) \cdot \{\mathbb{E}[e^{y_t} \cdot \mathbf{I}_{(-\infty, 0)}(y_t)] - \Pr(y_t < 0)\} \\ &= e^{\mu_t} - 1 + (\lambda - 1) \cdot \{e^{\mu_t} \cdot \Phi(c_1) - \Phi(c_2)\}, \end{aligned} \quad (9)$$

where

$$c_1 = \frac{-(\mu_x + \frac{1}{2}\sigma_x^2)\sqrt{t}}{\sigma_x}$$

and

$$c_2 = \frac{-(\mu_x - \frac{1}{2}\sigma_x^2)\sqrt{t}}{\sigma_x}.$$



Now, consider a buy-and-hold portfolio consisting of stocks and bonds. The price processes are

$$dS = \mu_s S dt + \sigma_s S dz_{s,t}$$

and

$$dB = \mu_b B dt + \sigma_b B dz_{b,t},$$

where

$$\begin{pmatrix} z_{s,t} \\ z_{b,t} \end{pmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, t \cdot \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right).$$

The value of a 50-50, buy-and-hold portfolio is

$$P_t = \frac{1}{2} e^{S_t} + \frac{1}{2} e^{B_t}.$$

Wilkinson suggests (see Levy [5] )

$$P_t = e^{w_t}$$

with

$$w_t \sim N(\alpha_t, \nu_t^2).$$

The moment generating function for  $P_t$  is given by

$$\begin{aligned} \mathbb{E} \left[ (e^{w_t})^k \right] &= \mathbb{E} [e^{kw_t}] \\ &= \mathbb{E} [\exp(k\alpha_t + \frac{1}{2}k^2\nu_t^2)] \end{aligned}$$

The first two moments,  $k = 1$  and  $k = 2$ , provide simultaneous equations in the unknown  $\alpha_t$  and  $\nu_t^2$  yielding

$$\alpha_t = 2 \cdot \log(\mathbb{E}[e^{w_t}]) - \frac{1}{2} \cdot \log(\mathbb{E}[e^{2w_t}]) \quad (10)$$

and

$$\nu_t^2 = \log(\mathbb{E}[e^{2w_t}]) - 2 \cdot \log(\mathbb{E}[e^{w_t}]). \quad (11)$$

For a 50-50, buy-and-hold portfolio, we have

$$\begin{aligned} \mathbb{E}[e^{w_t}] &= \mathbb{E} \left[ \frac{1}{2} e^{S_t} + \frac{1}{2} e^{B_t} \right] \\ &= \frac{1}{2} e^{\mu_s t} + \frac{1}{2} e^{\mu_b t} \end{aligned}$$

and

$$\mathbb{E}[e^{2w_t}] = \mathbb{E} \left[ \frac{1}{4} e^{2S_t} + \frac{1}{2} e^{S_t+B_t} + \frac{1}{4} e^{2B_t} \right],$$

with

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{4} e^{2S_t} \right] &= \frac{1}{4} \exp \left( 2 \left( \mu_s - \frac{1}{2} \sigma_s^2 \right) t + 2 \sigma_s^2 t \right) \\ &= \frac{1}{4} \exp \left( 2 \left( \mu_s + \frac{1}{2} \sigma_s^2 \right) t \right), \\ \mathbb{E} \left[ \frac{1}{4} e^{2B_t} \right] &= \frac{1}{4} \exp \left( 2 \left( \mu_b + \frac{1}{2} \sigma_b^2 \right) t \right), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{2} e^{S_t+B_t} \right] &= \frac{1}{2} \exp \left( \left( \mu_s + \mu_b - \left( \frac{1}{2} \sigma_s^2 + \frac{1}{2} \sigma_b^2 \right) \right) t + \frac{1}{2} \left( \sigma_s^2 + \sigma_b^2 + 2 \rho \sigma_s \sigma_b \right) t \right) \\ &= \frac{1}{2} \exp \left( \left( \mu_s + \mu_b + \rho \sigma_s \sigma_b \right) t \right). \end{aligned}$$

Inserting these expressions for  $\mathbb{E}[e^{w_t}]$  and  $\mathbb{E}[e^{2w_t}]$  in equation (10) and (11) gives  $\alpha_t$  and  $\nu_t^2$ .

Because

$$w_t \sim N \left( \alpha_t, \nu_t^2 \right),$$

we have, compare equation (9),

$$\mathbb{E} [v (e^{w_t} - 1)] = \exp(c_0) - 1 + (\lambda - 1) \cdot \{ \exp(c_0) \cdot \Phi(c_1) - \Phi(c_2) \},$$

where

$$\begin{aligned} c_0 &= \alpha_t + \frac{1}{2} \nu_t^2, \\ c_1 &= \frac{-(\alpha_t + \nu_t^2)}{\nu_t}, \end{aligned}$$

and

$$c_2 = \frac{-\alpha_t}{\nu_t}.$$

and the prospective value of the return on the portfolio is a function of  $t$  and  $\lambda$ , given

$\mu_s, \mu_b, \sigma_s, \sigma_b$ , and  $\rho$ .

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